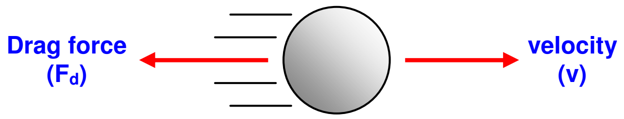
- THE SCIENCE OF SHOOTING ACADEMIC SERIES (PART 2) -

THE EFFECT OF WIND DRAG IN 1-D

We saw in the last article that the motion of a projectile can, in theory anyway, be described by a perfect parabola. In reality though, this is not the case. Indeed, the 2D path of a projectile subject to air drag cannot be described by any equation, such is the difficulty of the mathematics governing its motion. Keep this in mind.

It'd probably be best to start off with an explanation of exactly how air / wind / aerodynamic drag works. And just to make things a little easier, we'll stick to one-dimensional motion for the minute.

Aerodynamic drag always acts exactly in the opposite direction to which the projectile is moving. This is pretty obvious. Let's say we've got a projectile moving towards the right.



As a result of this motion, we have air drag pushing the projectile backwards towards the left, slowing it down.

The equation governing the drag, which is the same as that for any situation in which aerodynamic drag is present, is not so obvious however. It is:

$$F_D = \frac{1}{2} C_D \rho S v^2 \qquad \textit{... Eqn.1}$$

where:

	F_D	=	aerodynamic drag force (Newtons)
	C_D	=	coefficient of drag (no units)
	ρ	=	air density (kg/m^3)
	S	=	projectile cross-sectional area (m^2)
	v	=	projectile velocity (ms^{-1})

As you can see, wind drag is directly proportional to the square of its velocity. That means if we take the velocity of our projectile 'v' and triple it (say), the wind drag increases by a factor of 3^2 = 9. In short, the faster our projectile moves, the harder Mother Nature tries to push it back.

There are other, more expensive implications from this squared relationship as well.

Power usage, for example, is the result of drag multiplied by velocity. So for a plane moving through the air (say) the amount of power it needs to push through the air is the drag force 'F_D' multiplied by the velocity 'v', ie.

$$Power\ (watts) = \frac{1}{2} C_D \rho S v^3$$

Now the implication here is that if our pilot wants to increase the plane's speed from, say, 600kph to 700kph, the increase in power required to maintain this new speed is:

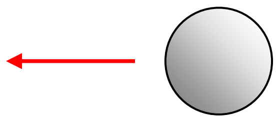
$$\left(\frac{700}{600}\right)^3 = 1.59$$

That means that if the pilot wants to increase the plane's speed by a paltry 17% he'll need 59% more power, and thus 59% more fuel (approximately), to do so!!!

Anyway, back on to the topic.

At this point we need to construct a free-body diagram of the projectile moving through the air. Keep in mind that we're ignoring gravity for the minute. This leaves us with only one force to consider.

$$F_D = \frac{1}{2} C_D \rho S v^2$$



Using Newton's second law (F = ma) we can construct the differential equation for the motion of the projectile. Note that 'm' is the mass of our projectile.

$$\sum F = m \cdot a = m \cdot \frac{dv}{dt}$$

$$\therefore \frac{dv}{dt} = \frac{\sum F}{m} = \frac{-\frac{1}{2} C_d \rho S v^2}{m}$$

(Note: the sign for the drag force is NEGATIVE here as it's pushing in the direction opposite to the velocity which, by convention, is positive)

Let's clean it all up:

$$\frac{dv}{dt} = -kv^2 \quad for \;\; k = \frac{C_d \rho S}{2m}$$

As we now want to solve this equation to obtain the function velocity (v) with respect to time (t), we have to rearrange it so that the like variables are together:

$$\frac{1}{v^2} \cdot dv = -k \cdot dt$$

This can now be solved. Keep in mind that v_o refers to the initial velocity of the projectile.

$$\int_{v_o}^{v} \left(\frac{1}{v^2}\right) dv = -k \int_0^t (1)\, dt$$

$$\left[\frac{-1}{v}\right]_{v_o}^{v} = -k[t]_0^t$$

$$\frac{-1}{v} + \frac{1}{v_o} = -kt$$

$$v(t) = \frac{v_o}{v_o kt + 1} \qquad \textit{... Eqn.2}$$

Perfect. That's exactly what we're after. This tells us that, as time progresses, the velocity of our projectile gradually decays.

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To put this into more meaningful terms, let's look at a single piece of 7.5 clay target shot (C_d = 0.42, Ø = 2.41mm, mass = 8.34×10^{-5} kg) which has been fired from a shotgun through the air (v_0 = 1250fps or 381ms^{-1}, ρ = 1.225 kgm^{-3}).

Now let's plug our values into Equation 2.

$$v(t) = \frac{v_o}{v_o kt + 1}$$

$$and \quad for \quad k = \frac{C_d \cdot \rho \cdot S}{2m}$$

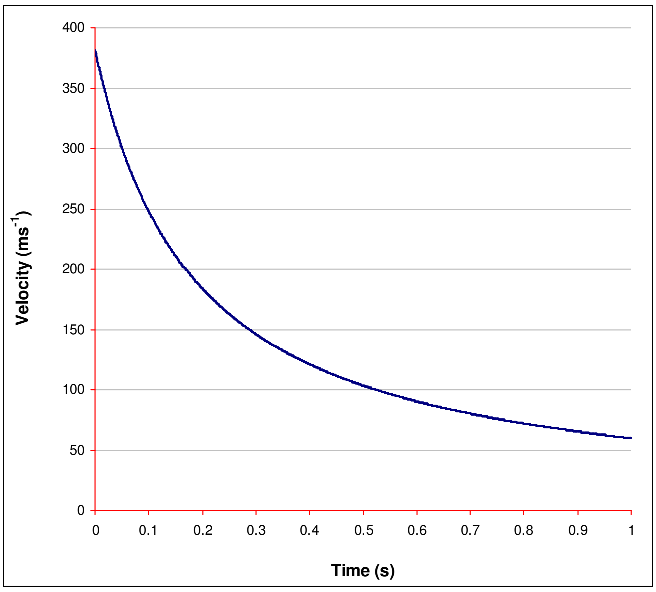
$$= \frac{0.42 \times 1.225 \times \left(4.56 \times 10^{-6}\right)}{2 \times \left(8.34 \times 10^{-5}\right)}$$

$$= 0.01407$$

$$thus \quad v(t) = \frac{381}{(381 \times 0.01407 \times t) + 1}$$

$$= \frac{381}{5.36t + 1}$$

If we plot this function, it looks something like this:



So as we can see, the velocity of our piece of shot starts at 381ms^{-1} then promptly heads southwards. Also note that its rate of slowing (ie. its decelleration) decreases as its velocity decreases. This makes sense because there's less wind drag on the shot when it's moving slower.

Actually it's even more intersting to note that the shot will drop below the sound barrier velocity (about 340ms^{-1}) almost as soon as it leaves the gun!

Knowing the velocity of the shot at any point in time may not be that useful a measure however. What some people really want to know is: "How long does it take for my shot to get out to the target?"

In order to work out displacement (x) as a function of time (t) we need to integrate our velocity function.
Let's first start with the definition of the velocity of our shot with respect to time, v(t).

$$v(t) = \frac{dx}{dt} = \frac{v_o}{v_o kt + 1}$$

Rearranging this D.E. give us:

$$dx = \frac{v_o}{v_o kt + 1} \, dt$$

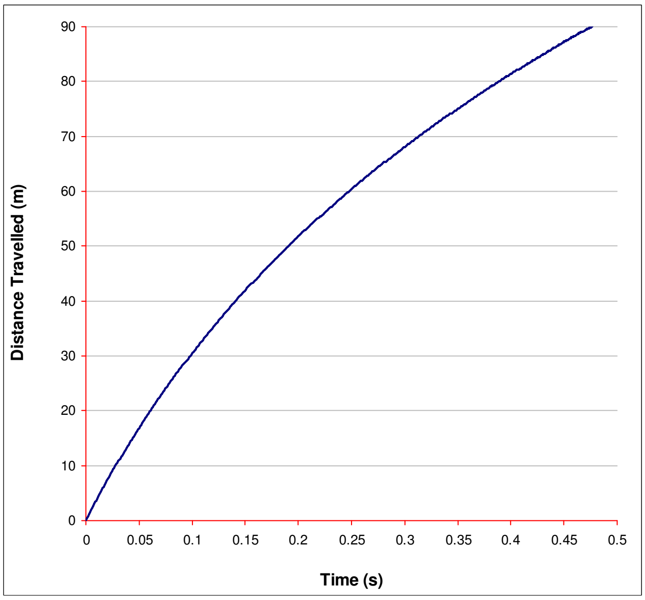
$$\int_0^x (1)\, dx = v_0 \cdot \int_0^t \frac{1}{v_0 kt + 1} \, dt$$

$$[x]_0^x = \frac{v_0}{v_0 k} \cdot \left[\log_e (v_0 kt + 1)\right]_0^t$$

$$x = \frac{1}{k} \cdot \left[\log_e (v_0 kt + 1) - 0\right]$$

$$and \; so \; \ldots \; x(t) = \frac{1}{k} \cdot \log_e (v_0 kt + 1) \qquad \textbf{... Eqn.3}$$

Now we all know pictures are often worth a thousand words, so let's graph our function of displacement (x) vs. time (t) so that we can see how long it takes for our shot to get out to our target.



Wow. It's really pretty. Actually it's not that exciting but there are a few very interesting features to note. These are:

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(1) The vertical axis showing the distance travelled doesn't extend past 90m as this is the maximum a clay target shot will ever travel (I mean EVER). Why? A ball trap target has a range limit of 75m a per the rule book, and if you include the 15m from you to the trap house, then ...

(2) The rate at which the shot gains distance seems to decrease as time progresses. This makes sense as the air resistance is contantly slowing the shot down.

(3) Yes. It will take you shot almost half a second to reach a ball trap target which is just about to hit the ground.

There are some more jovial things to note as well. Say you accidentally shoot the trap house. It takes your shot about 0.045 seonds to get there.

Sometimes it's not useful to have a graph of Velocity (v) vs. Time (t) though. Indeed, people usually want to know how fast the shot is travelling (v) at a certain distance (x) from them.

To work this out takes a little bit of mathematical gymnastics. The first thing we need to establish is knowing which parameter we want, being v(x), versus the parameters we've already got.

First off the rank is v(t). Seeing as though we've already got that one, let's look at what it actually is.

$$v(t) = \frac{dx}{dt} = \frac{v_0}{v_0 kt + 1}$$

Ok, that's not immediately useful to us. Instead, let's now have a look at the other parameters we know.

$$\frac{dv}{dt} = \frac{\Sigma F}{m} = \frac{-C_d \rho S v^2}{2m} = -kv^2$$

This isn't immediately useful to us either. How about we try using the differential chain rule on the parameters we have ...

$$\frac{dv}{dt} = \frac{dv}{dx} \times \frac{dx}{dt}$$

BINGO!!! This one is very, very useful. Why? Because we already know the first and third terms in the equation, and the middle term is only one integration away from the answer we want, ie. velocity as a function of distance travelled, v(x).

Plugging our other known equations into Equation 3 gives ...

$$-kv^2 = \frac{dv}{dx} \times v$$

This is now starting to look like something we'd actually want to work with. Another rearrangement should give us the D.E. which we can then solve.

$$\left(\frac{-1}{v}\right) dv = (k)\, dx$$

From here on in it's pretty much smiles and fluffy bunnies.

$$\int_{v_0}^{v} \left(\frac{1}{v}\right) dv = -(k) \int_0^x (1)\, dx$$

$$\left[\log_e(v)\right]_{v_0}^{v} = -k\,[x]_0^x$$

$$\log_e(v) - \log_e(v_0) = -kx$$

$$\log_e\left(\frac{v}{v_0}\right) = -kx$$

$$\frac{v}{v_0} = e^{-kx}$$

$$\text{thus } v(x) = v_0 e^{-kx} \qquad \textbf{\textit{... Eqn.4}}$$

And if you really want to go to the effort of substituting the true nature of 'k' back into the equation, you'll get:

$$v(x) = v_0 \cdot e^{-\frac{C_d \cdot \rho \cdot S}{2 \cdot m} x}$$

This is the really interesting relationship.

Yes, if we shoot the trap house it will take only 0.045sec for the shot to get there using x(t). What v(x) tells us is that when our shot actually hits the trap house, it'll still be travelling at about 308ms^{-1} which in more meaningful terms is about 1100kph. Now this might sound fast, but at this point the shot has actually lost about 35% of its kinetic energy or hitting power in layman's terms.

And let's just say that you can shoot the target when it's 15m from the trap house. At that point your shot will only be travelling at 250m/s or 900kph, which means that it will have lost 57% of its kinetic energy.

Yes. Shotguns are very, very shot range weapons.

Ok, so we haven't considered the effects of the supersonic and transonic regions so all of this is a bit, well, fanciful.

To be honest, we haven't even considered the problem that the coefficient of drag (C_d) is not necessarily constant over the range of velocities which a piece of shot may experience.

But it DOES show us that, in one dimension anyway, aerodynamic resistance will slow a piece of shot down with great gusto at first, but has less and less of an effect as the shot itself slows down.

Now let's get on with it and proceed to projectile motion in two dimensions, and if you thought 1D flight was difficult, it's probably a good thing you've no idea what's coming...

