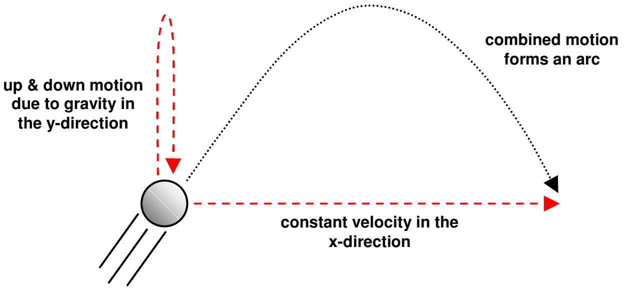
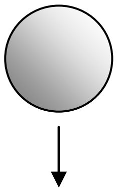
THE PATH OF A BULLET

It is known that, in the absence of air resistance (this will be dealt with in a later article), the vertical and horizontal motion of a projectile can be treated as being independent of each other.



So, if we focus solely on the vertical motion for the time being, the free body diagram of a projectile fired vertically with an initial velocity (v_0) is as follows:



g = 9.81ms^{-2}

The differential equation of vertical motion for the projectile, as it is subject to gravity ALONE, may be written as follows:

$$y''(t) = \frac{dv_y}{dt} = -g$$

To obtain our function for velocity as a function of time, we need to integrate the above equation. Note that as we want the answer to this integration to give us a velocity function *with respect to* (w.r.t) time rather than a definite number, we set the limits of integration as follows, noting that the initial velocity in the y-direction must be $v_{y,0}$ ms^{-1} and that the end time for our velocity function must indeed be a velocity (v). The starting time for our function on the time side of the equation is 0 sec. Obviously, as we want our equation to apply for ALL time, the end limit for the time integral is 't' itself.

$$\therefore \int_{v_{y,0}}^{v} (1)\,dv = \int_0^t (-g)\,dt$$

$$\therefore [v]_{v_{y,0}}^{v} = [-gt]_0^t$$

$$v - v_{y,0} = -gt$$

$$v_y(t) = -gt + v_{y,0}$$

Now recognising that...

$$v_y(t) = \frac{dy}{dt} = -gt + v_{y,0}$$

... we can obtain the function for the motion of the projectile in the vertical direction as a function of time, ie. y(t). Note that the initial height of release of the projectile is y_0:

$$\int_{y_0}^{y} (1)\,dy = \int_0^t \left(-gt + v_{y,0}\right) dt$$

$$[y]_{y_0}^{y} = \left[-\frac{1}{2}gt^2 + v_{y,0}t\right]_0^t$$

Evaluating this integral gives:

$$y - y_0 = -\frac{1}{2}gt^2 + v_{y,0}t$$

Which of course yields the equation we're interested in:

$$y(t) = -\frac{1}{2}gt^2 + v_{y,0}t + y_0 \qquad \text{... Eq.1}$$

So this is the equation of motion for the projectile in the vertical (y) direction. But what about the horizontal (x) direction?

It is known that a projectile travelling along a horizontal path, in the absence of wind resistance, has no net force acting on it and thus moves at constant velocity.

$$v_x(t) = \frac{dx}{dt} = v_{x,0}$$

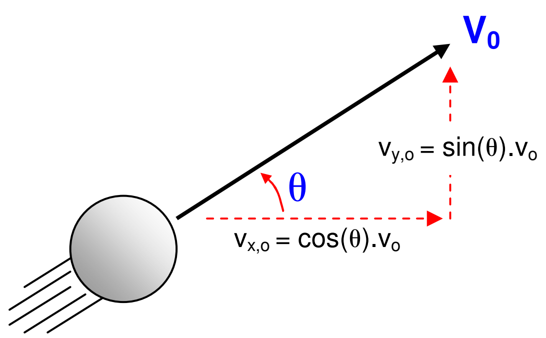
This of course gives us the position of the projectile in the x-direction if we integrate correctly. Assuming that the projectile is released from the position x = 0m, the equation for the velocity of the projectile in the x direction is thus:

$$\int_0^x (1)\,dx = \int_0^t \left(v_{x,0}\right) dt$$

$$[x]_0^x = \left[v_{x,0}t\right]_0^t$$

$$x(t) = t \cdot v_{x,0} \qquad \text{... Eq.2}$$

There's a complication in the fact that we don't yet know the initial vertical and horizontal velocities. To resolve this, consider the direction in which a projectile is fired:



This means that we need to modify both Equations 1 & 2 so that the correct initial velocities are shown. Let's rewrite the initial velocities for the projectile in each direction w.r.t. the absolute initial velocity of the projectile.

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$$y(t) = -\frac{1}{2}gt^2 + v_0\sin(\theta)t + y_0 \qquad \text{... Eq.3}$$

$$x(t) = t \cdot v_0\cos(\theta) \qquad \text{... Eq.4}$$

Hmmm. Whilst these look pretty, what we're really after is the path of the bullet in Cartesian coordinates (ie. x vs. y). What we've got are two seemingly independent equations, both with respect to time which we don't really care about.

Let's fix them up.

Our goal at this point is to get a single equation showing the vertical position of the projectile (y) as a function of its horizontal position (x) solely. To do this, we need to eliminate the variable of time (t).

Let's do a neat little trick. Take Equation 4 and rearrange it to make time (t) the subject of the equation, noting that *x(t)* is now rewritten simply as 'x' for shorthand purposes:

$$t = \frac{x}{v_0 \cdot \cos(\theta)}$$

Because all of the points of time in Equation 3 are the same as all of the points of time in Equation 4, we can substitute our new equation for time (above) into Equation 3. This gives:

$$y(t) = -\frac{1}{2}g\left(\frac{x}{v_0 \cdot \cos(\theta)}\right)^2 + v_0\sin(\theta)\cdot\left(\frac{x}{v_0 \cdot \cos(\theta)}\right) + y_0$$

Now after cleaning everything up and recognising that...

$$\frac{\sin(\theta)}{\cos(\theta)} = \tan(\theta)$$

... we finally reach the equation we've been after.

$$y(x) = \frac{-gx^2}{2v_0^2\cos^2(\theta)} + x\cdot\tan(\theta) + y_0 \qquad \text{... Eq.5}$$

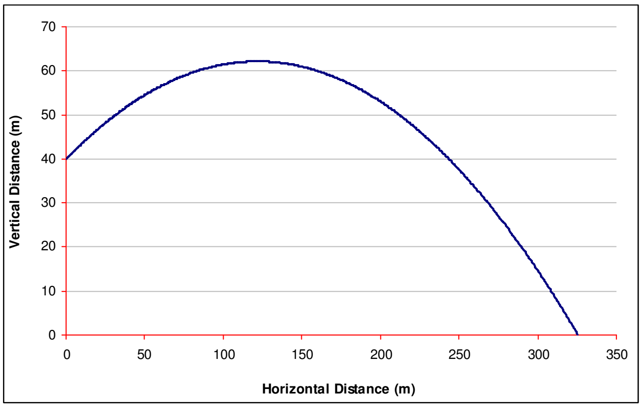
This equation is of course a negative parabola and is also the equation from which the notion that *all projectiles follow a parabolic path* comes from.

Now it's time for an example. Say we've got a soldier who's firing a mortar at 200fps (60.96ms^{-1}). He's standing on a hill 40m in the air and fires at an angle 20° from the horizontal.

As you'll see from the following graph (top of the next column) the projectile (mortar) manages to travel about 250m before it crashes into the ground. Further to this, it attains a maximum elevation just exceeding 60m.

Unfortunately though, this graph gives us no information about how much time the projectile spent in the air, at what distance away from the solider the projectile reached its maximum elevation, etc.

All in all, the graph is only part of the story.



Now having to read figures from a graph can be really laborious. Conveniently however, this is also where the fun starts as there are a whole heap of useful measures and parameters which naturally drop out of Equation 5.

For instance, the peak elevation occurs when the gradient of the projectile's flight is zero (ie. it's flying horizontally). So all we need to do is to take the derivative of our equation and set it equal to zero.

$$\frac{dy}{dx} = \frac{-gx}{v_0^2\cos^2(\theta)} + \tan(\theta)$$

$$0 = \frac{-gx}{v_0^2\cos^2(\theta)} + \tan(\theta)$$

$$\frac{gx}{v_0^2\cos^2(\theta)} = \tan(\theta)$$

$$x = \frac{\tan(\theta)\cdot\cos^2(\theta)\cdot v_0^2}{g}$$

$$x_{peak} = \frac{\sin(\theta)\cdot\cos(\theta)\cdot v_0^2}{g} \qquad \text{... Eq.6}$$

If we plug in our aforementioned parameters we find that the peak elevation of our projectile occurs at x = 121.7m

The corresponding elevation which the projectile reaches at this point can be deduced in either of two ways, being (1) take the figure of 121.7m and plug it into Equation 5 along with our other parameters, or (2) take Equation 6 and plug the whole thing into Equation 5 and simplify. This would look something like the following (skipping some steps):

$$y_{peak}(x) = \frac{-g\left(\frac{\sin(\theta)\cdot\cos(\theta)\cdot v_0^2}{g}\right)^2}{2v_0^2\cos^2(\theta)} + \left(\frac{\sin(\theta)\cdot\cos(\theta)\cdot v_0^2}{g}\right)\cdot\tan(\theta) + y_0$$

$$y_{peak}(x) = \frac{-\sin^2(\theta)\cdot v_0^2}{2g} + \frac{\sin^2(\theta)\cdot v_0^2}{g} + y_0$$

$$y_{peak}(x) = \frac{\sin^2(\theta)\cdot v_0^2}{2g} + y_0 \qquad \text{... Eq.7}$$

And that's about it for the peak elevation reached.

What about the amount of time it takes for the projectile to hit the ground? In the vertical direction, projectiles are subject ONLY to gravity. As an example, say you get a stone and drop it from shoulder height (1.5m). It will take about 0.4sec to fall to the ground. Projectiles behave no differently in the y-direction. Now get your gun, shoulder it and shoot along a dead-flat path (ie. perfectly horizontally). It too will take 0.4sec for the projectile to hit the ground. The point here is that the motion of the projectile in the x-direction has no effect on how long it takes for the projectile to crash into the ground.

On a practical basis, this means that if you're to shoot a target which is a long distance away, you actually need to aim above the target in order to give the projectile enough time to fall far enough down.

On a mathematical basis we can simply take Equation 1, set the equation to zero (ie. y(t) = 0) and solve for 't'. This will return two roots. One root may be negative or zero and should be ignored. The equation for the time a projectile takes to hit flat ground when fired from an elevation y_0 is:

$$t_{crash} = \frac{v_0 \sin(\theta) \pm \sqrt{v_0^2 \sin^2(\theta) + 2 g y_0}}{g} \qquad \text{... Eq.8}$$

And what about the location at which the projectile crashes into the ground? To obtain this, simply set our main equation (being Equation 5) to zero and solve for 'x'.

$$0 = \frac{-gx^2}{2v_0^2 \cos^2(\theta)} + x \cdot \tan(\theta) + y_0$$

$$x_{crash} = \frac{\sin(\theta) \cdot \cos(\theta) \cdot v_0^2}{g} \left(1 \pm \sqrt{1 + \frac{2 g y_0}{\sin^2(\theta) \cdot v_0^2}} \right) \qquad \text{... Eq.9}$$

Now you should obviously ignore the negative solution to this problem as that would imply that the projectile hits the ground behind the solder, which is of course utter rubbish. Indeed, if we use our parameters we find that the projectile crashes into the ground at about 325m.

Interestingly, if we assume that the projectile is fired from ground level (ie. y_0 = 0m) then Equation 9 simplifies down to the following (ignoring the trivial solution of x = 0m):

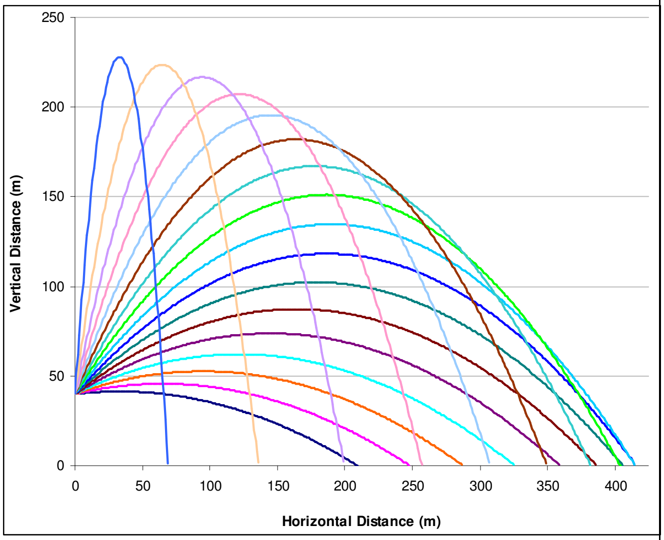
$$x_{crash} = \frac{2 \cdot \sin(\theta) \cdot \cos(\theta) \cdot v_0^2}{g}$$

This equation has its maximum value when $\theta = 45^\circ$ which supports the common misconception that the maximum range of all projectiles occurs when they are fired at this angle. Yes, this belief is correct when (and ONLY when) the projectile is fired from ground level. In reality, the elevation of release of the projectile has an effect which can't be ignored.

This brings up an interesting point. What is range? Now you've got to consider that not all targets will sit on the ground at an elevation of y = 0m. This is common sense. So how do we consider targets which don't sit on the ground?

And exactly how far can our mortar travel?

Let's proceed down this path and gauge the effect of altering the initial angle of aim. We'll use our soldier with his 200fps mortar again, but this time we'll plot all of the possible angles of firing from 0-90° in 5° increments.



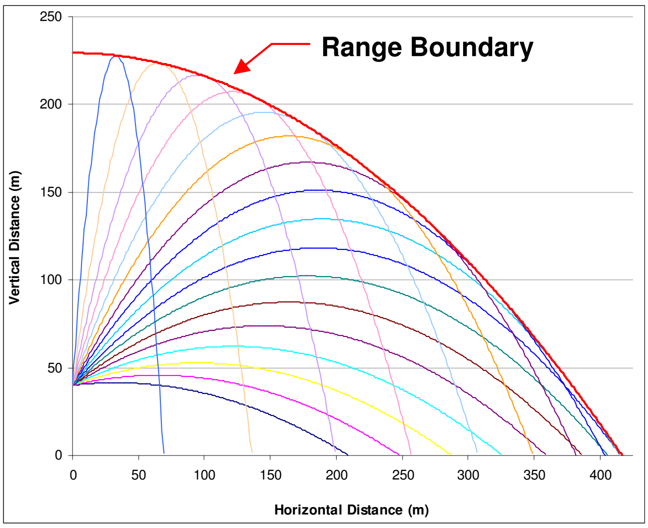
Interesting, eh? Ignoring all of the pretty coloured lines which even I can't help ogling at, the graph above shows two very, very important features. These are:

(1) There seems to be some sort of 'boundary' which the projectile can't get past. Could this be a good measure of range then?

(2) For every (x,y) location within this 'boundary' there are <u>two</u> projectile trajectories which will hit the target. The first of these is a high lofty trajectory, with the second being a more direct and flat trajectory.

Let's start with the first point. That imaginary boundary which the projectile can't seem to get past isn't imaginary at all; that's the range boundary of the projectile. The graph below (which is a repeat of the graph we saw before) shows the range boundary in red. There's also an equation for the range boundary, which is:

$$y(x) = \frac{-gx^2}{2v_0^2} + y_0 + \frac{v_0^2}{2g} \qquad \text{... Eq.10}$$



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It is impossible, no matter what angle the mortar is fired at, to hit any target which sits beyond the range boundary. Mathematically, this line depicts the path which would be followed if the soldier fired the mortar directly up in the air, then he (or she) climbed a ladder to the maximum height the previous mortar reached (in this case about 230m) and fired another mortar perfectly horizontally.

Now let's address our second point, being that every coordinate that falls within the range boundary can be hit with two different shots (a high lofty one, and a flatter one).

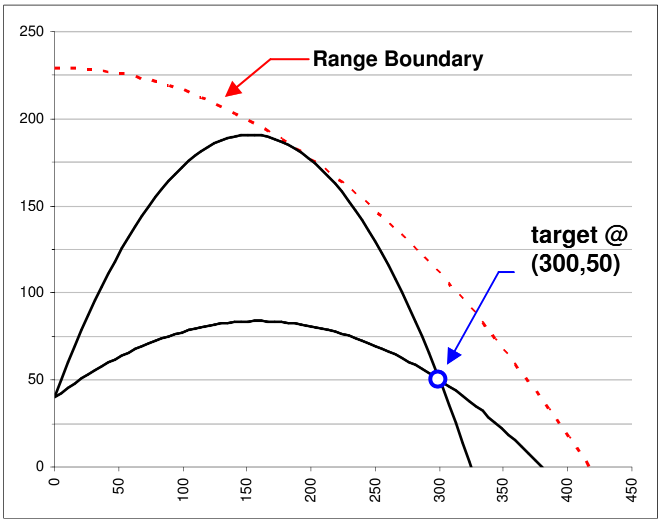
Now using the following trigonometric relationship...

$$\cos(\theta) = \frac{1}{\sqrt{1+\tan^2(\theta)}} \quad \text{for} \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$

... we can take our main equation, being Equation 5, plug in our (x,y) coordinates of a target we wish to hit and solve for our required firing angle(s), resulting in our two different trajectories:

$$\theta = \tan^{-1}\left(\frac{v_0^2 \pm \sqrt{v_0^4 + g\left(2v_0^2\left(y_0 - y\right) - gx^2\right)}}{gx} \right) \qquad \text{... Eq.11}$$

For example, the following graph shows our solider who's at an elevation of 40m with his 200fps mortar and he wants to hit a target 300m away from him at an elevation of 50m.



As you can see, there are two distinct angles (63.2° & 28.7°) at which the mortar can be fired in order to hit the target.

However there are a few complications. First and foremost is: if (x,y) doesn't lie within your range boundary, the equation above returns a meaningless answer.

The second thing to realise is that if you choose a set of target coordinates (x,y) such that the value of the square root is zero, then you get the same answer twice. This means your target is sitting precisely on the range boundary.

You should also note that neither of the paths ever manage to cross the range boundary. Just like the loftier trajectory, the flatter trajectory touches the range boundary off the graph but you can be assured that it will NEVER cross it.

Using this knowledge in combination with Equation 11, you can actually derive the equation which describes the range boundary itself (Equation 10) using the starting point of:

$$0 = \sqrt{v_0^4 + g\left(2v_0^2\left(y_0 - y\right) - gx^2\right)}$$

Now whilst a higher, loftier trajectory is useful if our solider wished to, say, fire over a building or a hill, there are consequences to this.

First and foremost is the amount of time it takes the mortar to reach its target. If we use a slightly modified version of Equation 8 to account for the elevation of our target...

$$t_{crash} = \frac{v_0 \sin(\theta) \pm \sqrt{v_0^2 \sin^2(\theta) + 2g\left(y_0 - y\right)}}{g}$$

... we see that it takes the mortar would take 10.9sec to hit the target, as opposed to the 5.6sec which would be taken if the flatter trajectory were to be adopted. This has implications for accuracy as a slight breeze has twice as much time to blow the mortar off course if the higher trajectory is used.

Obviously there are more parameters which could be calculated from the equations shown, however I at least hope that I've covered those measures most pertinent to the motion of projectiles.

Now for the sad part – none of this is realistic.

If we were on the moon, then yes, the equations I've given here apply. The reason for this is that there's no aerodynamic drag up there. Down here on earth however it's a different story; wind drag has a horrendous effect on projectile motion.

Worse still, there are no clearly defined analytic equations which govern the motion of projectiles. Well, none that I know of anyway. So far as I know, realistic projectile motion can only be estimated using numerical and computational methods.

So that's about it for the basics of projectile motion. I hope it's been useful to you. Next up is the effect of wind drag and aerodynamics, which are effectively one and the same.

References: none.

